

Supplemental Appendices for

A comparison of mean-variance efficiency tests

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B Computation of the asymptotic efficiency of the t -based PML estimator when the true distribution of the innovations is elliptical

To compute the efficiency of the t -based ML estimator relative to the GMM estimator under ellipticity of the innovations, we first need to compute the pseudo-true values of the parameters. For a fixed value of $\eta > 0$, we know that $\mathbf{a}_\infty(\eta) = \mathbf{a}_0$, $\mathbf{b}_\infty(\eta) = \mathbf{b}_0$ and $\boldsymbol{\Omega}_\infty(\eta) = \lambda_\infty^{-1}(\eta)\boldsymbol{\Omega}_0$, where $\lambda_\infty(\eta)$ solves

$$E \left[\frac{N\eta + 1}{1 - 2\eta + \eta\lambda_\infty(\eta)\varsigma} \frac{\lambda_\infty(\eta)\varsigma}{N} \middle| \boldsymbol{\phi}_0 \right] = 1, \quad (\text{B9})$$

with the expectation computed with respect to the true distribution of ς . This implicit equation is equivalent to the moment condition

$$E [\mathbf{s}_{\omega t}(\mathbf{a}_0, \mathbf{b}_0, \lambda_\infty^{-1}(\eta)\boldsymbol{\omega}_0, \eta) | \boldsymbol{\phi}_0] = \mathbf{0}$$

(see e.g. proof of Proposition 16 in Fiorentini and Sentana (2007)).

If η is not fixed, though, we will also have to compute the pseudo-true value of η , η_∞ , say. If the innovations are distributed as a platykurtic elliptical random vector, then we know from Proposition 4 that $\eta_\infty = 0$ and $\lambda_\infty(0) = 1$. But when the innovations are drawn from a leptokurtic elliptical random vector instead, then under standard regularity conditions η_∞ can be understood as the value that makes

$$E [s_{\eta t}(\boldsymbol{\theta}_\infty, \eta_\infty) | \boldsymbol{\phi}_0] = 0, \quad (\text{B10})$$

where

$$s_{\eta t}(\boldsymbol{\theta}, \eta) = \frac{\partial c(\eta)}{\partial \eta} + \frac{\partial g[\lambda_\infty \varsigma_t, \eta]}{\partial \eta}.$$

Fiorentini, Sentana and Calzolari (2003) show that for $\eta > 0$ this derivative is given by

$$\begin{aligned} \frac{\partial c(\eta)}{\partial \eta} &= \frac{N}{2\eta(1-2\eta)} - \frac{1}{2\eta^2} \left[\psi \left(\frac{N\eta+1}{2\eta} \right) - \psi \left(\frac{1}{2\eta} \right) \right], \\ \frac{\partial g(\varsigma_t, \eta)}{\partial \eta} &= -\frac{N\eta+1}{2\eta(1-2\eta)} \frac{\varsigma_t}{1-2\eta+\eta\varsigma_t} + \frac{1}{2\eta^2} \log \left[1 + \frac{\eta}{1-2\eta} \varsigma_t \right], \end{aligned}$$

where $\psi(\cdot)$ is the di-gamma or Gauss' psi function (see Abramovich and Stegun (1964)).

In general, the presence of a log term implies that we must compute (B10) by numerical integration using recursive adaptive Simpson quadrature, where the required expectation is taken with respect to the true distribution of ς .

Unfortunately, both $\partial g(\varsigma_t, \eta)/\partial \eta$ and especially $\partial c(\eta)/\partial \eta$ are numerically unstable for η small, as documented by Fiorentini, Sentana and Calzolari (2003). For that reason, we follow their

advice, and evaluate these expressions by means of the (directional) Taylor expansions around $\eta = 0$ in the following cases:

(i) if $\eta < 0.0008$, then use

$$\frac{\partial c_0(\eta)}{\partial \eta} = \frac{N(N+2)}{4} - \frac{N(N+2)(N-5)}{6}\eta + \frac{N(N+2)(N^2-6N+16)}{8}\eta^2$$

instead of $\partial c(\eta)/\partial \eta$, and

(ii) if $\eta < 0.03$ or $\eta\varsigma_t < 0.001$, then use

$$\begin{aligned} \frac{\partial g_0(\varsigma_t, \eta)}{\partial \eta} = & -\frac{N+2}{2}\varsigma_t + \frac{1}{4}\varsigma_t^2 \\ & + \left[-2(N+2)\varsigma_t + \frac{N+4}{2}\varsigma_t^2 - \frac{1}{3}\varsigma_t^3 \right] \eta \\ & + \left[-12(N+2)\varsigma_t + 6(N+3)\varsigma_t^2 - (N+6)\varsigma_t^3 + \frac{1}{8}\varsigma_t^4 \right] \frac{\eta^2}{2} \\ & + \left[\begin{aligned} & -96(N+2)\varsigma_t + 24(3N+8)\varsigma_t^2 - 24(N+4)\varsigma_t^3 \\ & + 3(N+8)\varsigma_t^4 - \frac{12}{5}\varsigma_t^5 \end{aligned} \right] \frac{\eta^3}{6} \\ & + \left[\begin{aligned} & -960(N+2)\varsigma_t + 600(2N+5)\varsigma_t^2 - 1440(3N+10)\varsigma_t^3 \\ & + 120(N+5)\varsigma_t^4 - 12(N+10)\varsigma_t^5 + 10\varsigma_t^6 \end{aligned} \right] \frac{\eta^4}{24} \end{aligned} \quad (\text{B11})$$

instead of $\partial g(\varsigma_t, \eta)/\partial \eta$. Consequently, we evaluate (B10) as the weighted average of this expectation conditional on the complementary events $\varsigma_t < 0.001\eta_0$ and $\varsigma_t > 0.001\eta_0$ weighted by the corresponding probabilities. In many cases, both the expected value of (B11) conditional on $\varsigma_t < 0.001\eta_0$ and $P(\varsigma_t < 0.001\eta_0|\phi_0)$ can be computed analytically.

Having obtained the pseudo-true values, then we need to compute

$$M_{II}^H[\eta, \lambda_\infty(\eta)] = E \left[\frac{N\eta + 1}{1 - 2\eta + \eta\lambda_\infty(\eta)\varsigma_t} \left(1 + \frac{2\eta}{1 - 2\eta + \eta\lambda_\infty(\eta)\varsigma_t} \frac{\lambda_\infty(\eta)\varsigma_t}{N} \right) \middle| \phi_0 \right] \quad (\text{B12})$$

and

$$M_{II}^O[\eta, \lambda_\infty(\eta)] = E \left[\left(\frac{N\eta + 1}{1 - 2\eta + \eta\lambda_\infty(\eta)\varsigma_t} \right)^2 \frac{\lambda_\infty(\eta)\varsigma_t}{N} \middle| \phi_0 \right]. \quad (\text{B13})$$

It turns out that we can obtain analytical expressions for these expectations in the two examples that we consider in the paper.

B.1 Kotz innovations

As discussed in section 2.1, ς is Gamma distributed when the true innovations follow a Kotz distribution. Consequently, (B9), (B12) and (B13) can be decomposed in terms of the form

$$a \cdot E \left[\left(\frac{1}{b + dy} \right)^k y^h \right],$$

where $y = \alpha\varsigma/N$ is distributed as a standardized Gamma with parameter $\alpha = N[(N+2)\kappa+2]^{-1}$, k and h are non-negative integers, and $a, b > 0$, and $d > 0$ are real constants. In fact we only need to find an analytical expression for $E[(1+cy)^{-k}]$ for $k = 1$ and $k = 2$, where $c = d/b > 0$, as

$$\frac{a}{b^k} E \left[\left(\frac{1}{1+cy} \right)^k y^h \right] = \frac{a}{b^k} \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)} E \left[\frac{1}{(1+cy^*)^k} \right],$$

where $\Gamma(a)$ is the complete Gamma function and y^* a standardized Gamma with parameter $\alpha+h$.

To do so, we first compute the moment generating function of $1+cy$, which is given by

$$M_{1+cy}(t) = E[e^{t(1+cy)}] = e^t E[e^{tcy}] = \frac{e^t}{(1-ct)^\alpha}$$

since $M_y(t) = E(e^{ty}) = (1-t)^{-\alpha}$. Then, we can exploit the result in equation (3) in Cressie, Davis, Folks and Policello (1981), which in our case yields

$$E \left[\frac{1}{(1+cy)^k} \right] = \frac{1}{\Gamma(k)} \int_0^\infty t^{k-1} M_{1+cy}(-t) dt$$

for any positive random variable y for which the above integral is well defined.

If we use the change of variable $s = t + c^{-1}$, so that $t = s - c^{-1}$, $cs = ct + 1$ and $ds = dc$, then we obtain that for $k = 1$,

$$E \left[\frac{1}{(1+cy)} \right] = \int_0^\infty \frac{e^{-t}}{(1+cy)^\alpha} dt = \frac{e^{c^{-1}}}{c^\alpha} \int_{c^{-1}}^\infty \frac{e^{-s}}{s^\alpha} ds = \frac{e^{c^{-1}}}{c^\alpha} \Gamma(1-\alpha, c^{-1}).$$

where $\Gamma(a, x)$ is the non-normalized incomplete Gamma function, which can be computed using standard software such as *Mathematica* or *Maple*. Similarly, for $k = 2$ we end up with

$$\begin{aligned} E \left[\frac{1}{(1+cy)^2} \right] &= \int_0^\infty t \frac{e^{-t}}{(1+cy)^\alpha} dt \\ &= \int_{c^{-1}}^\infty (s - c^{-1}) \frac{e^{-(s-c^{-1})}}{(cs)^\alpha} ds \\ &= \frac{e^{c^{-1}}}{c^\alpha} \left[\int_{c^{-1}}^\infty \frac{e^{-s}}{s^{\alpha-1}} ds - c^{-1} \int_{c^{-1}}^\infty \frac{e^{-s}}{s^\alpha} ds \right] \\ &= \frac{e^{c^{-1}}}{c^\alpha} [\Gamma(2-\alpha, c^{-1}) - c^{-1} \Gamma(1-\alpha, c^{-1})] \\ &= \frac{e^{c^{-1}}}{c^\alpha} \{ [(1-\alpha) - c^{-1}] \Gamma(1-\alpha, c^{-1}) \} + c^{-1}. \end{aligned}$$

Finally, note that the terms $E[\varsigma^k | \varsigma < 0.001\eta_0^{-1}; \phi_0]$ that appear in the expectation of (B11), together with $P[\varsigma < 0.001\eta_0^{-1} | \phi_0]$ can be easily computed in terms of incomplete Gamma functions too.

B.2 Two-component scale mixture of normals

Since in this case ς is $\text{Gamma}(N/2, 1/2)$ conditional on the realization of the mixing variable s , we can use exactly the same formulas as in the case of the Kotz distribution, and then average across the two values of s . For instance,

$$M_{II}^H[\eta, \lambda_\infty(\eta)] \equiv \pi E \left[\frac{N\eta + 1}{1 - 2\eta + \eta\lambda_\infty(\eta)\varpi y} \left(1 + \frac{2\eta}{1 - 2\eta + \eta\lambda_\infty(\eta)\varpi y} \frac{\lambda_\infty(\eta)\varpi y}{N} \right) \middle| \phi_0, s = 1 \right] \\ + (1 - \pi) E \left[\frac{N\eta + 1}{1 - 2\eta + \eta\lambda_\infty(\eta)\varpi \kappa y} \left(1 + \frac{2\eta}{1 - 2\eta + \eta\lambda_\infty(\eta)\varpi \kappa y} \frac{\lambda_\infty(\eta)\varpi \kappa y}{N} \right) \middle| \phi_0, s = 0 \right],$$

where $\varpi\alpha y/N$ is distributed as a standardised Gamma with parameter $\alpha = N/2$.

C EM recursions for the multivariate t distribution

In this Appendix we specialise the expressions in Appendices B and D of Mencía and Sentana (2008) to the conditionally homoskedastic multivariate regression model with symmetric t innovations that we are considering. The rationale for using the EM algorithm comes from the fact that the model $\mathbf{r}_t = \mathbf{a} + \mathbf{b}r_{Mt} + \boldsymbol{\Omega}^{1/2}\boldsymbol{\varepsilon}_t^*$, with $\boldsymbol{\varepsilon}_t^*|r_{Mt}, I_{t-1}; \phi_0 \sim i.i.d. t(\mathbf{0}, \mathbf{I}_N, \nu_0)$ can be rewritten as

$$\mathbf{r}_t = \mathbf{a} + \mathbf{b}r_{Mt} + \boldsymbol{\Omega}^{1/2} \sqrt{\frac{\nu_0 - 2}{\xi_t}} \boldsymbol{\varepsilon}_t^\circ$$

where $\boldsymbol{\varepsilon}_t^\circ|\xi_t, r_{Mt}, I_{t-1}; \phi_0 \sim N(\mathbf{0}, \mathbf{I}_N)$ and $\xi_t|\phi_0 \sim \text{Gamma}(\nu_0/2, 1/2)$.

Given that we know $f(\mathbf{r}_t|\xi_t, r_{Mt}; \phi)$, $f(\xi_t|\phi)$ and $f(\mathbf{r}_t|r_{Mt}; \phi)$, we can use Bayes theorem to obtain the distribution of ξ_t conditional on $\mathbf{r}_t$ and r_{Mt} . Specifically,

$$f(\xi_t|\mathbf{r}_t, r_{Mt}; \phi) = f(\mathbf{r}_t|\xi_t, r_{Mt}; \phi)f(\xi_t|\phi)/f(\mathbf{r}_t|r_{Mt}; \phi) \propto f(\mathbf{r}_t|\xi_t, r_{Mt}; \phi)f(\xi_t|\phi).$$

Straightforward algebra shows that we can write

$$f(\xi_t|\mathbf{r}_t, r_{Mt}; \phi) \propto \xi_t^{N/2} \exp \left[-\frac{\varsigma_t}{2} \frac{\eta}{1 - 2\eta} \xi_t \right] \xi_t^{\frac{1}{2\eta} - 1} \exp \left(-\frac{\xi_t}{2} \right) \\ \propto \xi_t^{\frac{N\eta+1}{2\eta} - 1} \exp \left[-\frac{\xi_t}{2} \left(\frac{\eta\varsigma_t}{1 - 2\eta} + 1 \right) \right]$$

where $\varsigma_t = (\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt})'\boldsymbol{\Omega}^{-1}(\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt})$, so that

$$\xi_t|\mathbf{r}_t, r_{Mt}; \phi \sim \text{Gamma} \left\{ \frac{N\eta + 1}{2\eta}, \frac{1}{2} \left[1 + \frac{\eta\varsigma_t}{1 - 2\eta} \right] \right\}.$$

On this basis, we can show that the EM recursions with respect to $\mathbf{a}$, $\mathbf{b}$ and $\boldsymbol{\omega}$ will be given by

$$\begin{pmatrix} \mathbf{a}^{(i+1)} \\ \mathbf{b}^{(i+1)} \end{pmatrix} = \left\{ \left[\sum_{s=1}^T \xi_{s|s}^{(i)} \begin{pmatrix} 1 & r_{Ms} \\ r_{Ms} & r_{Ms}^2 \end{pmatrix} \right]^{-1} \otimes \mathbf{I}_N \right\} \sum_{t=1}^T \left\{ \left[\xi_{t|t}^{(i)} \begin{pmatrix} 1 \\ r_{Mt} \end{pmatrix} \right] \otimes \mathbf{r}_t \right\}$$

and

$$\boldsymbol{\omega}^{(i+1)} = \text{vech} \left[\frac{1}{T} \frac{\tilde{\eta}^{(i)}}{1 - 2\tilde{\eta}^{(i)}} \sum_{t=1}^T \xi_{t|t}^{(i)} (\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt}) (\mathbf{r}_t - \mathbf{a} - \mathbf{b}r_{Mt})' \right],$$

where

$$\xi_{t|t}^{(i)} = E[\xi_t | \mathbf{r}_t, r_{Mt}; \mathbf{a}^{(i)}, \mathbf{b}^{(i)}, \boldsymbol{\omega}^{(i)}, \tilde{\eta}^{(i)}] = \frac{N\tilde{\eta}^{(i)} + 1}{\tilde{\eta}^{(i)}} \left[\frac{\tilde{\eta}^{(i)} \varsigma_t}{1 - 2\tilde{\eta}^{(i)}} + 1 \right]^{-1}.$$

Although it is also possible to use the EM principle to update η , it involves numerical optimisation, so in practice it may be better to define $\tilde{\eta}^{(i+1)} = \arg \max L_T(\tilde{\boldsymbol{\theta}}^{(i+1)}, \eta)$ using $\tilde{\eta}^{(i)}$ as starting value. To initialise the EM recursions, we use the $\hat{\boldsymbol{\theta}}_{GMM}$ and the sequential ML estimator for η , $\hat{\eta}_{SML}$, which in turn we obtain using the MM estimator (26) as starting value.

D The information matrix for scale mixtures of normals

The density of ς when ε^* is a two-component scale mixture of normals is

$$h(\varsigma; \boldsymbol{\eta}) = \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \varsigma^{N/2-1} \left[\pi \exp\left(-\frac{1}{2\varpi}\varsigma\right) + (1 - \pi)\varkappa^{-N/2} \exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right) \right],$$

where $\varpi = [\pi + \varkappa(1 - \pi)]^{-1}$. If we combine $h(\varsigma; \boldsymbol{\eta})$ with expression (2.21) in Fang, Kotz and Ng (1990), then (5) follows. Hence,

$$\begin{aligned} \text{M}_h(\boldsymbol{\eta}) &= E \left[\delta^2(\varsigma; \boldsymbol{\eta}) \frac{\varsigma}{N} \middle| \boldsymbol{\phi} \right] \\ &= \int_0^\infty \frac{1}{\varpi^2} \left\{ \pi + (1 - \pi)\varkappa^{-N/2} \exp\left[-\frac{1 - \varkappa}{2\varpi\varkappa}\varsigma\right] \right\}^{-1} \\ &\quad \times \left\{ \pi^2 + 2\pi(1 - \pi)\varkappa^{-(N/2+1)} \exp\left[-\frac{1 - \varkappa}{2\varpi\varkappa}\varsigma\right] \right. \\ &\quad \left. + (1 - \pi)^2 \varkappa^{-(N+2)} \exp\left[-\frac{1 - \varkappa}{\varpi\varkappa}\varsigma\right] \right\} \\ &\quad \times \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{\varsigma^{N/2}}{N} \exp\left(-\frac{1}{2\varpi}\varsigma\right) d\varsigma \\ &= \text{A}_1 + \text{A}_2 + \text{A}_3, \end{aligned}$$

where

$$\text{A}_1 = \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} \pi^2 \int_0^\infty \left\{ \pi + (1 - \pi)\varkappa^{-N/2} \exp\left[-\frac{1 - \varkappa}{2\varpi\varkappa}\varsigma\right] \right\}^{-1} \frac{\varsigma^{N/2}}{N} \exp\left(-\frac{1}{2\varpi}\varsigma\right) d\varsigma,$$

$$\begin{aligned} \text{A}_2 &= \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} 2\pi(1 - \pi) \int_0^\infty \left\{ \pi + (1 - \pi)\varkappa^{-N/2} \exp\left[-\frac{1 - \varkappa}{2\varpi\varkappa}\varsigma\right] \right\}^{-1} \\ &\quad \times \varkappa^{-(N/2+1)} \frac{\varsigma^{N/2}}{N} \exp\left(-\frac{1}{2\varpi\varkappa}\varsigma\right) d\varsigma \end{aligned}$$

and

$$\begin{aligned} A_3 &= \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} (1-\pi)^2 \int_0^\infty \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi \varkappa} \varsigma \right] \right\}^{-1} \\ &\quad \times \varkappa^{-(N+2)} \frac{\varsigma^{N/2}}{N} \exp \left[-\frac{2-x}{2\varpi \varkappa} \varsigma \right] d\varsigma. \end{aligned}$$

By analogy with Masoom and Nadarajah (2007), we can use the change of variable $v = \frac{1}{2\varpi \varkappa} (1-\varkappa) \varsigma$, so that $d\varsigma = 2\varpi \varkappa (1-\varkappa)^{-1} dv$, whence we get

$$\begin{aligned} A_1 &= \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} \frac{1}{N} \pi \left(\frac{2\varpi \varkappa}{1-\varkappa} \right)^{N/2+1} \\ &\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp(-v) \right\}^{-1} v^{N/2} \exp \left(-\frac{\varkappa}{1-\varkappa} v \right) dv \\ &= \frac{1}{\varpi} \pi \left(\frac{\varkappa}{1-\varkappa} \right)^{N/2+1} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{\varkappa}{1-\varkappa} \right), \end{aligned}$$

$$\begin{aligned} A_2 &= \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} 2(1-\pi) \frac{\varkappa^{-(N/2+1)}}{N} \left(\frac{2\varpi \varkappa}{1-\varkappa} \right)^{N/2+1} \\ &\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp(-v) \right\}^{-1} v^{N/2} \exp \left(-\frac{1}{1-\varkappa} v \right) dv \\ &= \frac{1}{\varpi} 2(1-\pi) \left(\frac{1}{1-\varkappa} \right)^{N/2+1} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{1}{1-\varkappa} \right) \end{aligned}$$

and

$$\begin{aligned} A_3 &= \frac{(2\varpi)^{-N/2}}{\varpi^2 \Gamma(N/2)} \frac{(1-\pi)^2}{\pi} \frac{\varkappa^{-(N+2)}}{N} \left(\frac{2\varpi \varkappa}{1-\varkappa} \right)^{N/2+1} \\ &\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp(-v) \right\}^{-1} v^{N/2} \exp \left[-\frac{2-\varkappa}{1-\varkappa} v \right] dv \\ &= \frac{1}{\varpi} \frac{(1-\pi)^2}{\pi} [\varkappa(1-\varkappa)]^{-(N/2+1)} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{2-\varkappa}{1-\varkappa} \right), \end{aligned}$$

where $F(z, s, r)$ denotes the Lerch function (see Erdelyi, 1981), which can be represented as

$$F(z, s, r) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{v^{s-1} \exp(-rv)}{1 - z \exp(-v)} dv.$$

This function can be accurately computed using standard software such as *Mathematica*.

Therefore,

$$\begin{aligned} M_{ll}(\boldsymbol{\eta}) &= \frac{1}{\varpi} \pi \left(\frac{\varkappa}{1-\varkappa} \right)^{N/2+1} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{\varkappa}{1-\varkappa} \right) \\ &\quad + \frac{2}{\varpi} (1-\pi) \left(\frac{1}{1-\varkappa} \right)^{N/2+1} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{1}{1-\varkappa} \right) \\ &\quad + \frac{1}{\varpi} \frac{(1-\pi)^2}{\pi} [\varkappa(1-\varkappa)]^{-(N/2+1)} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{2-\varkappa}{1-\varkappa} \right). \end{aligned}$$

Similarly, we can use

$$\begin{aligned}
\frac{\partial \delta(\varsigma; \boldsymbol{\eta})}{\partial \varsigma} &= -\frac{1-\kappa}{2\varpi^2\kappa} \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\}^{-1} \\
&\quad \times (1-\pi)\kappa^{-(N/2+1)} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \\
&\quad + \frac{1-\kappa}{2\varpi^2\kappa} \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\}^{-2} \\
&\quad \times \left\{ \pi + (1-\pi)\kappa^{-N/2+1} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\} \\
&\quad \times (1-\pi)\kappa^{-N/2} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right]
\end{aligned}$$

to compute $M_{ss}(\boldsymbol{\eta})$ from

$$M_{ss}(\boldsymbol{\eta}) = E \left[\frac{2\partial \delta[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]}{\partial \varsigma} \frac{\varsigma_t(\boldsymbol{\theta})}{N} \middle| \boldsymbol{\phi} \right] + 1,$$

with

$$\begin{aligned}
E \left[\frac{2\partial \delta[\varsigma; \boldsymbol{\eta}]}{\partial \varsigma} \frac{\varsigma^2}{N(N+2)} \middle| \boldsymbol{\phi} \right] &= \int_0^\infty \frac{\varsigma^2}{N(N+2)} \left\{ \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\}^{-1} \right. \\
&\quad \times \frac{(1-\kappa)}{\varpi^2\kappa} (1-\pi)\kappa^{-N/2} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \\
&\quad \times \left\{ \pi + (1-\pi)\kappa^{-(N/2+1)} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\} \\
&\quad \left. - \frac{(1-\kappa)}{\varpi^2\kappa} (1-\pi)\kappa^{-(N/2+1)} \exp \left[-\frac{(1-\kappa)}{2\varpi\kappa} \varsigma \right] \right\} \\
&\quad \times \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \varsigma^{N/2-1} \exp \left(-\frac{1}{2\varpi} \varsigma \right) d\varsigma \\
&= B_1 + B_2 + B_3
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= -\frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} (1-\pi)(1-\kappa)\kappa^{-(N/2+2)} \int_0^\infty \varsigma^{N/2+1} \exp \left[-\frac{1}{2\varpi\kappa} \varsigma \right] d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} (1-\pi)(1-\kappa)\kappa^{-(N/2+2)} (2\varpi\kappa)^{(N/2+2)} \Gamma \left(\frac{N}{2} + 2 \right) \\
&= -(1-\pi)(1-\kappa)
\end{aligned}$$

$$\begin{aligned}
B_2 &= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} \pi(1-\pi)(1-\kappa)\kappa^{-(N/2+1)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2+1} \exp\left[-\frac{1}{2\varpi\kappa}\varsigma\right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} (1-\pi)(1-\kappa)\kappa^{-(N/2+1)} \left(\frac{2\varpi\kappa}{1-\kappa}\right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi}\kappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2+1} \exp\left[-\frac{1}{1-\kappa}v\right] dv \\
&= (1-\pi)\kappa \left(\frac{1}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi}\kappa^{-N/2}, \frac{N}{2}+2, \frac{1}{1-\kappa}\right)
\end{aligned}$$

and

$$\begin{aligned}
B_3 &= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} (1-\pi)^2(1-\kappa)\kappa^{-(N+2)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2+1} \exp\left[-\frac{2-\kappa}{2\varpi\kappa}\varsigma\right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{(N+2)\Gamma(N/2)} \frac{1}{N\varpi^2} \frac{(1-\pi)^2}{\pi} (1-\kappa)\kappa^{-(N+2)} \left(\frac{2\varpi\kappa}{1-\kappa}\right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi}\kappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2+1} \exp\left[-\frac{2-\kappa}{1-\kappa}v\right] dv \\
&= \frac{(1-\pi)^2}{\pi} \kappa^{-N/2} \left(\frac{1}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi}\kappa^{-N/2}, \frac{N}{2}+2, \frac{2-\kappa}{1-\kappa}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
M_{ss}(\boldsymbol{\eta}) &= -(1-\kappa)(1-\pi) \\
&\quad + (1-\pi) \left(\frac{1}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi}\kappa^{-N/2}, \frac{N}{2}+2, \frac{1}{1-\kappa}\right) \\
&\quad + \frac{(1-\pi)^2}{\pi} \kappa^{-N/2} \left(\frac{1}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi}\kappa^{-N/2}, \frac{N}{2}+2, \frac{2-\kappa}{1-\kappa}\right).
\end{aligned}$$

Finally, we can use

$$\begin{aligned}
\frac{\partial\delta(\varsigma; \boldsymbol{\eta})}{\partial\pi} &= \varpi(1-\kappa)\delta(\varsigma; \boldsymbol{\eta}) \\
&\quad + \frac{1}{\varpi} \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \\
&\quad \left\{ 1 + \left[\frac{\varsigma}{2}(1-\pi)(1-\kappa)^2\kappa^{-(N/2+2)} - \kappa^{-(N/2+1)}\right] \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\} \\
&\quad - \frac{1}{\varpi} \left\{ 1 + \left[\frac{\varsigma}{2}(1-\pi)(1-\kappa)^2\kappa^{-(N/2+1)} - \kappa^{-N/2}\right] \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\} \\
&\quad \times \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\}^{-2} \\
&\quad \times \left\{ \pi + (1-\pi)\kappa^{-(N/2+1)} \exp\left[-\frac{(1-\kappa)}{2\varpi\kappa}\varsigma\right] \right\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \delta(\varsigma; \boldsymbol{\eta})}{\partial \varkappa} &= \varpi(1 - \pi)\delta(\varsigma; \boldsymbol{\eta}) \\
&- \left[\left(\frac{N}{2} + 1 \right) (1 - \pi) \varkappa^{-(N/2+2)} + \frac{\varsigma}{2} [1 - \pi(1 - \varkappa^{-2})] (1 - \pi) \varkappa^{-(N/2+1)} \right] \\
&\times \frac{1}{\varpi} \left\{ \pi + (1 - \pi) \varkappa^{-N/2} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\}^{-1} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \\
&+ \left[\frac{N}{2} (1 - \pi) \varkappa^{-(N/2+1)} + \frac{\varsigma}{2} [1 - \pi(1 - \varkappa^{-2})] (1 - \pi) \varkappa^{-N/2} \right] \\
&\times \frac{1}{\varpi} \left\{ \pi + (1 - \pi) \varkappa^{-N/2} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\}^{-2} \\
&\times \left\{ \pi + (1 - \pi) \varkappa^{-(N/2+1)} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right]
\end{aligned}$$

to compute

$$\begin{aligned}
M_{sr}(\boldsymbol{\eta}) &= -E \left[\frac{\varsigma_t(\boldsymbol{\theta})}{N} \frac{\partial \delta[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]}{\partial \boldsymbol{\eta}'} \middle| \boldsymbol{\phi} \right] \\
&= -E \left[\frac{\varsigma_t(\boldsymbol{\theta})}{N} \left(\frac{\partial \delta[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]}{\partial \pi}, \frac{\partial \delta[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]}{\partial \varkappa} \right) \middle| \boldsymbol{\phi} \right].
\end{aligned}$$

We then need

$$\begin{aligned}
E \left[\frac{\varsigma}{N} \frac{\partial \delta(\varsigma, \boldsymbol{\eta})}{\partial \pi} \middle| \boldsymbol{\phi} \right] &= \int_0^\infty \frac{\varsigma}{N} \left\{ (1 - \varkappa) \left[\pi + (1 - \pi) \varkappa^{-(N/2+1)} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right] \right. \\
&+ \frac{1}{\varpi} \left\{ 1 + \left[\frac{\varsigma}{2} (1 - \pi) (1 - \varkappa)^2 \varkappa^{-(N/2+2)} - \varkappa^{-(N/2+1)} \right] \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\} \\
&- \left\{ \pi + (1 - \pi) \varkappa^{-N/2} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\}^{-1} \\
&\times \frac{1}{\varpi} \left\{ 1 + \left[\frac{\varsigma}{2} (1 - \pi) (1 - \varkappa)^2 \varkappa^{-(N/2+1)} - \varkappa^{-N/2} \right] \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right\} \\
&\times \left[\pi + (1 - \pi) \varkappa^{-(N/2+1)} \exp \left[-\frac{1 - \varkappa}{2\varpi \varkappa} \varsigma \right] \right] \Bigg\} \\
&\times \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \varsigma^{N/2-1} \exp \left(-\frac{1}{2\varpi} \varsigma \right) d\varsigma \\
&= C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N} \left[(1 - \varkappa) \pi + \frac{1}{\varpi} \right] \int_0^\infty \varsigma^{N/2} \exp \left(-\frac{1}{2\varpi} \varsigma \right) d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{(2\varpi)^{N/2+1}}{N} \left[(1 - \varkappa) \pi + \frac{1}{\varpi} \right] \Gamma \left(\frac{N}{2} + 1 \right) \\
&= \varpi \pi (1 - \varkappa) + 1
\end{aligned}$$

$$\begin{aligned}
C_2 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} [\varpi(1-\pi)(1-\kappa) - 1] \kappa^{-(N/2+1)} \int_0^\infty \varsigma^{N/2} \exp\left(-\frac{1}{2\varpi\kappa}\varsigma\right) d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{(2\varpi)^{N/2+1}}{N\varpi} [\varpi(1-\pi)(1-\kappa) - 1] \Gamma\left(\frac{N}{2} + 1\right) \\
&= \varpi(1-\pi)(1-\kappa) - 1
\end{aligned}$$

$$\begin{aligned}
C_3 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} (1-\pi)(1-\kappa)^2 \kappa^{-(N/2+2)} \int_0^\infty \varsigma^{N/2+1} \exp\left(-\frac{1}{2\varpi\kappa}\varsigma\right) d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} (1-\pi)(1-\kappa)^2 (2\varpi)^{N/2+2} \Gamma\left(\frac{N}{2} + 2\right) \\
&= \varpi(1-\pi)(1-\kappa)^2 \left(\frac{N}{2} + 1\right)
\end{aligned}$$

$$\begin{aligned}
C_4 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{\pi}{N\varpi} \int_0^\infty \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{1-\kappa}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2} \exp\left(-\frac{1}{2\varpi}\varsigma\right) d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} \left(\frac{2\varpi\kappa}{1-\kappa}\right)^{N/2+1} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \kappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp\left(-\frac{\kappa}{1-\kappa}v\right) dv \\
&= -\left(\frac{\kappa}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi} \kappa^{-N/2}, \frac{N}{2} + 1, \frac{\kappa}{1-\kappa}\right)
\end{aligned}$$

$$\begin{aligned}
C_5 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} [(1-\pi) - \pi\kappa] \kappa^{-(N/2+1)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{1-\kappa}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2} \exp\left[-\frac{1}{2\varpi\kappa}\varsigma\right] d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} \left[\frac{1-\pi}{\pi} - \kappa\right] \kappa^{-(N/2+1)} \left(\frac{2\varpi\kappa}{1-\kappa}\right)^{N/2+1} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \kappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp\left[-\frac{1}{1-\kappa}v\right] dv \\
&= -\left[\frac{1-\pi}{\pi} - \kappa\right] \left(\frac{1}{1-\kappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi} \kappa^{-N/2}, \frac{N}{2} + 1, \frac{1}{1-\kappa}\right)
\end{aligned}$$

$$\begin{aligned}
C_6 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} \frac{\pi(1-\pi)}{2} \kappa^{-(N/2+1)} (1-\kappa)^2 \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\kappa^{-N/2} \exp\left[-\frac{1-\kappa}{2\varpi\kappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2+1} \exp\left[-\frac{1}{2\varpi\kappa}\varsigma\right] d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} \frac{(1-\pi)}{2} \kappa^{-(N/2+1)} (1-\kappa)^2 \left(\frac{2\varpi\kappa}{1-\kappa}\right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \kappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp\left[-\frac{1}{1-\kappa}v\right] dv \\
&= -\varpi(1-\pi)\kappa \left(\frac{1}{1-\kappa}\right)^{N/2} \left(\frac{N}{2} + 1\right) F\left(-\frac{1-\pi}{\pi} \kappa^{-N/2}, \frac{N}{2} + 2, \frac{1}{1-\kappa}\right)
\end{aligned}$$

$$\begin{aligned}
C_7 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{N\varpi} (1-\pi) \varkappa^{-(N+1)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\}^{-1} \varsigma^{N/2} \exp \left[-\frac{2-\varkappa}{2\varpi\varkappa} \varsigma \right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1-\pi}{\pi} \frac{\varkappa^{-(N+1)}}{N\varpi} \left(\frac{2\varpi\varkappa}{1-\varkappa} \right)^{N/2+1} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp \left[-\frac{2-\varkappa}{1-\varkappa} v \right] dv \\
&= \frac{1-\pi}{\pi} \varkappa^{-N/2} \left(\frac{1}{1-\varkappa} \right)^{N/2+1} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{2-\varkappa}{1-\varkappa} \right);
\end{aligned}$$

$$\begin{aligned}
C_8 &= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} (1-\pi)^2 (1-\varkappa)^2 \varkappa^{-(N+2)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\}^{-1} \varsigma^{N/2+1} \exp \left[-\frac{2-\varkappa}{2\varpi\varkappa} \varsigma \right] d\varsigma \\
&= -\frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} \frac{(1-\pi)^2}{\pi} (1-\varkappa)^2 \varkappa^{-(N+2)} \left(\frac{2\varpi\varkappa}{1-\varkappa} \right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2+1} \exp \left[-\frac{2-\varkappa}{1-\varkappa} v \right] dv \\
&= -\varpi \frac{(1-\pi)^2}{\pi} \varkappa^{-N/2} \left(\frac{1}{1-\varkappa} \right)^{N/2} \left(\frac{N}{2} + 1 \right) F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 2, \frac{2-\varkappa}{1-\varkappa} \right);
\end{aligned}$$

and

$$\begin{aligned}
E \left[\frac{\varsigma_t(\boldsymbol{\theta})}{N} \frac{\partial \delta[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]}{\partial \varkappa} \middle| \boldsymbol{\phi} \right] &= \int_0^\infty \frac{\varsigma}{N} \left\{ (1-\pi) \left[\pi + (1-\pi) \varkappa^{-(N/2+1)} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right] \right. \\
&\quad \left. - \left[\left(\frac{N}{2} + 1 \right) (1-\pi) + \frac{\varsigma}{2} [1 - \pi(1-\varkappa^{-2})] \varkappa \right] \right. \\
&\quad \left. \times \frac{1}{\varpi} \varkappa^{-(N/2+2)} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right. \\
&\quad \left. + \left[\frac{N}{2} (1-\pi) + \frac{\varsigma}{2} [1 - \pi(1-\varkappa^{-2})] (1-\pi) \varkappa \right] \frac{\varkappa^{-(N/2+1)}}{\varpi} \right. \\
&\quad \left. \times \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\}^{-1} \right. \\
&\quad \left. \times \left\{ \pi + (1-\pi) \varkappa^{-(N/2+1)} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\} \\
&\quad \times \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \varsigma^{N/2-1} \exp \left(-\frac{1}{2\varpi} \varsigma \right) d\varsigma \\
&= D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \frac{(2\varpi)^{-N/2} (1-\pi)\pi}{\Gamma(N/2) N} \int_0^\infty \varsigma^{N/2} \exp\left(-\frac{1}{2\varpi}\varsigma\right) d\varsigma \\
&= \frac{(2\varpi)^{-N/2} (1-\pi)\pi}{\Gamma(N/2) N} (2\varpi)^{N/2+1} \Gamma\left(\frac{N}{2} + 1\right) \\
&= \varpi(1-\pi)\pi \\
D_2 &= -\frac{(2\varpi)^{-N/2} \varkappa^{-(N/2+2)}}{\Gamma(N/2) N} (1-\pi) \left[\frac{1}{\varpi} \left(\frac{N}{2} + 1\right) - (1-\pi)\varkappa \right] \int_0^\infty \varsigma^{N/2} \exp\left(\frac{1}{2\varpi\varkappa}\varsigma\right) d\varsigma \\
&= -\frac{(2\varpi)^{-N/2} \varkappa^{-(N/2+2)}}{\Gamma(N/2) N} (1-\pi) \left[\frac{1}{\varpi} \left(\frac{N}{2} + 1\right) - (1-\pi)\varkappa \right] (2\varpi\varkappa)^{N/2+1} \Gamma\left(\frac{N}{2} + 1\right) \\
&= -(1-\pi) \frac{1}{\varkappa} \left[\left(\frac{N}{2} + 1\right) - (1-\pi)\varkappa\varpi \right] \\
D_3 &= -\frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2N\varpi} [1 - \pi(1 - \varkappa^{-2})] (1-\pi) \varkappa^{-(N/2+1)} \int_0^\infty \varsigma^{N/2+1} \exp\left(\frac{1}{2\varpi\varkappa}\varsigma\right) d\varsigma \\
&= -\frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2N\varpi} [1 - \pi(1 - \varkappa^{-2})] (1-\pi) \varkappa^{-(N/2+1)} (2\varpi\varkappa)^{N/2+2} \Gamma\left(\frac{N}{2} + 2\right) \\
&= -\left(\frac{N}{2} + 1\right) \varpi(1-\pi)\varkappa [1 - \pi(1 - \varkappa^{-2})] \\
D_4 &= \frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2\varpi} (1-\pi)\pi \varkappa^{-(N/2+1)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\varkappa^{-N/2} \exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2} \exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2\varpi} (1-\pi)\varkappa^{-(N/2+1)} \left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+1} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp\left[-\frac{1}{1-\varkappa}v\right] dv \\
&= \frac{N}{2} (1-\pi) \left(\frac{1}{1-\varkappa}\right)^{N/2+1} F\left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{1}{1-\varkappa}\right), \\
D_5 &= \frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2N\varpi} \pi(1-\pi) [1 - \pi(1 - \varkappa^{-2})] \varkappa^{-N/2} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi)\varkappa^{-N/2} \exp\left[-\frac{1-\varkappa}{2\varpi\varkappa}\varsigma\right] \right\}^{-1} \varsigma^{N/2+1} \exp\left[-\frac{1}{2\varpi\varkappa}\varsigma\right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2} 1}{\Gamma(N/2) 2N\varpi} (1-\pi) [1 - \pi(1 - \varkappa^{-2})] \varkappa^{-N/2} \left(\frac{2\varpi\varkappa}{1-\varkappa}\right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2+1} \exp\left[-\frac{1}{1-\varkappa}v\right] dv \\
&= \varpi(1-\pi) [1 - \pi(1 - \varkappa^{-2})] \left(\frac{\varkappa}{1-\varkappa}\right)^2 \left(\frac{1}{1-\varkappa}\right)^{N/2} \\
&\quad \times \left(\frac{N}{2} + 1\right) F\left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 2, \frac{1}{1-\varkappa}\right),
\end{aligned}$$

$$\begin{aligned}
D_6 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{\varpi} \frac{1}{2} (1-\pi)^2 \varkappa^{-(N+2)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\}^{-1} \varsigma^{N/2} \exp \left[-\frac{2-\varkappa}{2\varpi\varkappa} \varsigma \right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2\varpi} \frac{(1-\pi)^2}{\pi} \varkappa^{-(N+2)} \left(\frac{2\varpi\varkappa}{1-\varkappa} \right)^{N/2+1} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2} \exp \left[-\frac{2-\varkappa}{1-\varkappa} v \right] dv \\
&= \frac{N(1-\pi)^2}{2} \frac{1}{\pi} [\varkappa(1-\varkappa)]^{-(N/2+1)} F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 1, \frac{2-\varkappa}{1-\varkappa} \right),
\end{aligned}$$

and

$$\begin{aligned}
D_7 &= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} [1 - \pi(1 - \varkappa^{-2})] (1-\pi)^2 \varkappa^{-(N+1)} \\
&\quad \times \int_0^\infty \left\{ \pi + (1-\pi) \varkappa^{-N/2} \exp \left[-\frac{1-\varkappa}{2\varpi\varkappa} \varsigma \right] \right\}^{-1} \varsigma^{N/2+1} \exp \left[-\frac{2-\varkappa}{2\varpi\varkappa} \varsigma \right] d\varsigma \\
&= \frac{(2\varpi)^{-N/2}}{\Gamma(N/2)} \frac{1}{2N\varpi} \frac{(1-\pi)^2}{\pi} [1 - \pi(1 - \varkappa^{-2})] \varkappa^{-(N+1)} \left(\frac{2\varpi\varkappa}{1-\varkappa} \right)^{N/2+2} \\
&\quad \times \int_0^\infty \left\{ 1 + \frac{1-\pi}{\pi} \varkappa^{-N/2} \exp[-v] \right\}^{-1} v^{N/2+1} \exp \left[-\frac{2-\varkappa}{1-\varkappa} v \right] dv \\
&= \left(\frac{N}{2} + 1 \right) \varpi \frac{(1-\pi)^2}{\pi} [1 - \pi(1 - \varkappa^{-2})] \varkappa^{-(N/2-1)} \left(\frac{1}{1-\varkappa} \right)^{N/2+2} \\
&\quad \times F \left(-\frac{1-\pi}{\pi} \varkappa^{-N/2}, \frac{N}{2} + 2, \frac{2-\varkappa}{1-\varkappa} \right),
\end{aligned}$$

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